Bayesian Inverse Estimation
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UQ for Stochastic System
\[
\frac{\partial u}{\partial t} (t, x, Z) = \mathcal{L}(u) + \text{boundary/initial conditions}
\]
\[
u(t, x, Z): [0, T] \times \mathbb{D} \times \mathbb{R}^n \mapsto \mathbb{R}^n
\]

- Uncertain inputs are characterized by \( n_z \) random variables \( Z \)
- Successful UQ simulation requires specifications of the random inputs: distribution function, correlation, etc.

- **Quantification of Input Uncertainty can be difficult**
  - Experiments can be expensive and inflexible
  - Direct measurement of the input parameters is often not possible

- **Objective: Inversely estimate properties of input parameters**
  - Data: measurement of computable quantities --- data \( d \)
  - Prior estimate of input parameters (by intuition if necessary)
  - Simulation: forward computations of the measured quantities: \( G(Z) \) -- forward problem

- **Bayesian Approach:**
  - \( d = G(Z) + e, \quad e \in \mathbb{R}^n \) is i.i.d.
  - Data = forward model + error

\[
\pi(Z \mid d) = \frac{\pi(d \mid Z) \pi(Z)}{\int \pi(d \mid Z) \pi(Z) \, dZ}
\]

- **Challenge** for Bayesian approach: Computational cost \( \Rightarrow \) Sampling requires repetitive simulations of the system

- **Generalized Polynomial chaos based Bayesian Approach:**

  - \( u_n(t, x, Z) = \sum_{|\mathcal{K}|=n} \hat{u}_n(t, x) \phi_n(Z) \), \# of basis = \( n_z + N \)

  - **Theorem [Marzouk and Xiu, 2009]:**
    - Convergence of posterior distribution is as fast as the convergence if the forward problem stochastic solver.

  - Application to Burgers’ equation: Estimate of boundary condition
    - Red: convergence of forward problem; Blue: convergence of distribution

  - Application to freely vibrating cantilever damping

Gas damping of fixed-free microcantilevers is simulated using a finite volume method. The damping factor is found to be most sensitive to its thickness \( t \), vibration frequency \( f \) and gas density \( \rho \). A response surface for the damping factor is developed based on second-order generalized polynomial chaos which significantly reduces the cost of the forward solution.