Uncertainty Propagation in MEMS
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Introduction

Sources of uncertainties in MEMS
- Low cost manufacturing processes
- Residual stresses
- Irregular surface topography
- Chemical contamination

Electrostatic MEMS devices

Uncertain parameters in MEMS:
- Material properties: Young’s modulus, etc.
- Geometrical features: Dimensions, gap, etc.
- Operating environment and boundary conditions.

Young’s modulus for polysilicon
[Clark, Ph.D. thesis, 2005]
Mathematical Models

Mechanical Analysis

\[ \rho \ddot{u} = \nabla \cdot (FS) \quad \text{in } \Omega \]
\[ u = G \quad \text{on } d\Omega_g \]
\[ P \cdot N = H \quad \text{on } d\Omega_h \]
\[ u \big|_{t=0} = G_0 \quad \text{in } \Omega \]
\[ \dot{u} \big|_{t=0} = V_0 \quad \text{in } \Omega \]

Electrostatic Analysis

\[ \phi(p) = \int_{d\omega} G(p, q) \sigma(q) d\gamma_q + C \]
\[ C_T = \int_{d\omega} \sigma(q) d\gamma_q \]
\[ P_e = \frac{\sigma^2}{2\varepsilon} \]

Fluidic Analysis

\[ \nabla \cdot \left( (1 + 6K) h^3 P_f \nabla P_f \right) = 12\eta \frac{\partial (P_f h)}{\partial t} \quad \text{in } \Omega_f \]

Deterministic coupled problem

\[ \mathcal{L}(u, \sigma, P_f; x, t) = 0, \quad x \in \Omega \]

Deformation of a MEM beam under electrostatic and fluid pressure

De and Aluru, JMM, 2006

Coupling term

\[ H = J P F^{-T} N \]
\[ P_t = P_e - P_{f_e} \]
\[ P_{f_e} = \frac{1}{W} \int_{\Omega_f} P_f dZ \]
## Galerkin vs Collocation Methods

<table>
<thead>
<tr>
<th>Stochastic Galerkin method</th>
<th>Stochastic collocation method</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Advantages:</strong></td>
<td><strong>Advantages:</strong></td>
</tr>
<tr>
<td>□ Fast convergence for smooth solutions</td>
<td>□ Easy to implement (non-intrusive)</td>
</tr>
<tr>
<td>□ Sensitivities are automatically computed</td>
<td>□ Completely decoupled (parallelization)</td>
</tr>
<tr>
<td>□ Offers faster convergence than sampling based approaches such as MC</td>
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</tr>
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</table>

| **Disadvantages:** | **Disadvantages:** |
| □ Results in coupled equations | □ Performance deteriorates in the presence of discontinuities |
| □ Intrusive: substantial code development. | □ Results in coupled equations |
| □ Numerical implementation is not easy for nonlinearities, multi-physics etc | □ Intrusive: substantial code development. |
| □ Employs global basis functions and hence not suitable for capturing discontinuities. | □ Numerical implementation is not easy for nonlinearities, multi-physics etc |

Adaptive stochastic collocation

Stochastic Collocation Framework

The coupled MEMS problem was given as: $\mathcal{L} (u, \sigma, P_f; x, t, \xi) = 0$

Collocation procedure:

- Substitute the interpolation for $(u, \sigma, P_f)$ in the governing equation.

$$
\left[ \hat{u}(\cdot), \hat{\sigma}(\cdot), \hat{P}_f(\cdot) \right] = \sum_{j=1}^{N} u \left[ \xi(\cdot), \xi(\cdot), \xi(\cdot) \right] L_\xi(\cdot)
$$

- Collocation at $\xi^k$, $k = 1, \ldots, N$ gives

$$
\mathcal{L} \left( \hat{u}, \hat{\sigma}, \hat{P}_f; x, t, \xi^k \right) = 0 \quad \forall k \in 1, \ldots, N
$$

- Using the property of the interpolating functions $L_i(\xi^k) = \delta_{ik}$, we get

$$
\mathcal{L} \left( \hat{u}^k, \hat{\sigma}^k, \hat{P}_f^k; \xi, x, t, \xi^k \right) = 0 \quad \forall k \in 1, \ldots, N
$$

Stochastic Collocation Method

Solving $N$ deterministic problems at each nodal points $\{\xi^k, k = 1, \ldots, N\}$
Stochastic Collocation Framework

Consequences
- Equivalent to solving the deterministic system at $N$ discrete points
- The entire stochastic solution can be constructed by sampling at these discrete points
- The resulting equations are naturally decoupled (scope of parallelization)

Computation of statistics
$$\mu(\hat{u}) = E[\hat{u}] \quad \nu(\hat{u}) = E[(\hat{u} - \mu(\hat{u}))^2]$$

Question: How do we construct the multivariate interpolation effectively?

$$\hat{u}(\xi) \equiv I \ u(\xi) = \sum_{j=1}^{N} u(\xi^j)L_j(\xi)$$
Support nodes $\Theta_N = \{\xi^j\}_{j=1}^{N} \subset \Gamma$

Smolyak Algorithm: constructs multi-dimensional interpolants based on sparse grids
- Univariate interpolation
- Univariate to multivariate interpolation
Univariate Interpolation

Given \( f:[0,1] \rightarrow \mathbb{R} \), which is approximated as,

\[
I^k(f)(\xi) = \sum_{j=1}^{m_k} f(\xi_j^k) l_j^k(\xi) \quad \text{for each } k \geq 1
\]

\[ l_j^k(\xi_i^k) = 0 \quad \forall i \neq j \]

Support nodes: \( \chi^k = \left\{ \xi_j^k \mid \xi_j^k \in [0,1], j = 1, \ldots, m_k \right\} \)

Basis functions: \( I^k = \left\{ l_j^k \mid l_j^k \in C[0,1], j = 1, \ldots, m_k \right\} \)

Choice of basis functions and support nodes

- Piecewise linear basis functions with equidistant nodes, or,
- Lagrange basis functions with Chebyshev-Gauss-Lobatto nodes, etc.

The 1D support nodes are nested:

\[ \chi^k \subset \chi^{k+1}, k \geq 1 \]
Univariate to Multivariate Interpolation

Given a function \( f: [0,1]^n \rightarrow \mathbb{R} \), we need to construct a multi-dimensional Interpolation formula. Assume the 1D interpolation formula is given as

\[
I^k(f) = \sum_{\xi_j^k \in \mathcal{X}^k} f\left(\xi_j^k\right)_{j}^k
\]

Tensor product formula (using nodal 1D formula)

\[
I(f) \equiv \left( I^{k_1} \otimes \cdots \otimes I^{k_n} \right)(f) = \sum_{\xi_j^{k_1} \in \mathcal{X}^{k_1}} \cdots \sum_{\xi_j^{k_n} \in \mathcal{X}^{k_n}} f\left(\xi_j^{k_1}, \ldots, \xi_j^{k_n}\right) \left( I^{k_1}_{j_1} \otimes \cdots \otimes I^{k_n}_{j_n} \right)
\]

Example: Nodes in each direction \( m_i = 2 \forall i \Rightarrow N = 2^n \)

Extremely expensive for higher dimensions!
Smolyak Algorithm: Sparse Grids

**Basic Idea:** Reduce the number of support nodes while maintaining the quality of the interpolation formula (up to a logarithmic factor).

**Smolyak construction**

- The sparse grid interpolation of \( f \) is given as,

\[
I \left( f \right) = A_{q,n} \left( f \right) = \sum_{q+1 \leq |k| \leq q+n} (-1)^{q+n-|k|} \cdot \binom{n-1}{q+n-|k|} \cdot \left( I_{k_1} \otimes \cdots \otimes I_{k_n} \right) \left( f \right)
\]

- Depth of interpolation \( q \geq 0 \)

- To compute \( A_{q,n} \left( f \right) \), evaluate \( f \) at *sparse grid *

\[
\Theta_N = H_{q,n} = \bigcup_{q+1 \leq |k| \leq q+n} \left( \chi^{k_1} \otimes \cdots \otimes \chi^{k_n} \right)
\]

Tensor vs Sparse Grids

depth $q=3$

$N = 81$

depth $q=4$

$N = 289$

depth $q=5$

$N = 1089$

$N = 29$

$N = 65$

$N = 145$
Hierarchical Univariate Interpolation

Motivation
- Nested grids
- Yields error estimators, used for adaptive procedure

Define $I^0 = 0$, and the incremental interpolant $\Delta^k = I^k - I^{k-1}, \forall k \geq 1$

Noting that $f(\xi^k_j) - I^{k-1}(f)(\xi^k_j) = 0 \quad \forall \xi^k_j \in \chi^{k-1}$

Hierarchical surpluses: $w^k_j$

Hierarchical 1D interpolation

$\Delta^k (f) = \sum_{j=1}^{m^k} \frac{f(\xi^k_j) - I^{k-1}(f)(\xi^k_j)}{w^k_j}$

Support nodes:

$\chi^k = \bigcup_{i=1}^{k} \chi^i \Delta \quad \chi^k \Delta = \chi^k \setminus \chi^{k-1}$

[A. Klimke and B. Wholmuth 2005]
Hierarchical Smolyak Algorithm

Smolyak construction

- Given the 1D hierarchical interpolant

\[ I^k(f) = I^{k-1}(f) + \Delta^k(f) = \sum_{i=1}^{k} \Delta^i(f) \]

- The sparse grid interpolation of \( f \) is given as,

\[ A_{q,n}(f) = \sum_{|k| \leq n+q} \left( \Delta^{k_1} \otimes \cdots \otimes \Delta^{k_n} \right)(f) \]

Depth of interpolation \( q \)
\[ |k| = k_1 + \cdots + k_n \]
\[ q \geq 0 \]

- The sparse interpolant can be written in a hierarchical manner, with

\[ A_{q,n}(f) = A_{q-1,n}(f) + \sum_{|k| = n+q} \left( \Delta^{k_1} \otimes \cdots \otimes \Delta^{k_n} \right)(f) \]
\[ \Delta A_{q,n}(f) = 0 \]

We can increase the accuracy of interpolation without having to discard the earlier results.
Smolyak Algorithm: Sparse Grids

The $n$-dimensional incremental sparse interpolant $\Delta A_{q,n}(f)$ can be written as:

$$\Delta A_{q,n}(f) = \sum_{|k|=n+q} \sum_{j} \left( \prod_{i=1}^{n} I_{j_i}^{k_i} \right) \left( \prod_{i=1}^{n} \left( f(\xi_{j_i}^{k_i}, ..., \xi_{j_n}^{k_n}) - A_{q-1,n}(f)(\xi_{j_i}^{k_i}, ..., \xi_{j_n}^{k_n}) \right) \right)$$

Hierarchical surpluses

To construct the interpolant $A_{q,n}$, we only need to compute the function value at the sparse grid nodes given by

$$H_{q,n} = \bigcup_{|k| \leq n+q} \left( \chi_{\Delta}^{k_1} \times \cdots \times \chi_{\Delta}^{k_n} \right)$$

$$\chi_{\Delta}^{k} = \chi^{k} \setminus \chi^{k-1}$$

Since the 1D nodes are nested $\chi^{k-1} \subset \chi^{k}$, the sparse grids can also be written in a hierarchical manner as

$$H_{q,n} = H_{q-1,n} \cup \Delta H_{q,n}$$

where

$$\Delta H_{q,n} = \bigcup_{|k|=n+q} \left( \chi_{\Delta}^{k_1} \times \cdots \times \chi_{\Delta}^{k_n} \right)$$

S. Smolyak, Quadrature and interpolation formulas for tensor products of certain classes of functions, Soviet Math. Dokl., (1963)
Example: Smolyak Algorithm

$H_{2,2}$

$|k| = 2$

$|k| = 3$

$|k| = 4$
Interpolation of Non-Smooth Functions

\[ f(\xi_1, \xi_2) = \begin{cases} 
0, & \text{if } \xi_1 > \alpha_1, \xi_2 > \alpha_2, \\
\sin(\pi \xi_1) \sin(\pi \xi_2), & \text{otherwise},
\end{cases} \]

For \( \alpha_1 = \alpha_2 = 0.5 \), \( f \) has sharp discontinuities at \( \xi_1 = 0.5, \xi_2 = 0.5 \).

Evolution of interpolated function and corresponding sparse grids using multi-linear basis functions.
Interpolation of Non-Smooth Functions

Interpolation error using linear and polynomial basis functions for the discontinuous function.

Error in mean and variance using multi-linear basis functions.

<table>
<thead>
<tr>
<th>Linear</th>
<th>Polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>Error</td>
</tr>
<tr>
<td>Smooth</td>
<td>3329</td>
</tr>
<tr>
<td>Non-smooth</td>
<td>69633</td>
</tr>
</tbody>
</table>

Interpolation Error: smooth vs. non-smooth functions.
Adaptive Sparse Grid Collocation

Classical Smolyak construction increases the depth $q$ of the interpolation, in order to increase the accuracy.

$$A_{q,n}(f) = A_{q-1,n}(f) + \Delta A_{q,n}(f) = \sum_{|k|=n+q} (\Delta^k \otimes \ldots \otimes \Delta^k)(f)$$

Issues with classical framework:
- Lagrange interpolation: Global basis functions cannot capture discontinuities
- Piecewise linear interpolation: All products satisfying $|k| = n + q$ are selected

Basic Idea: Locally approximate the stochastic solution with high accuracy, by adaptively decomposing the random domain into subdomains.

Adaptive stochastic collocation
- Within each subdomain, employ hierarchical surpluses to estimate local contribution towards global error, and,
- more sensitive random dimensions
- Based on the adaptivity criterion, subdivide the random element along most sensitive random dimensions
Adaptivity Criterion

Criterion 1
\[ \beta_s = \max(w_2, w_3, w_4, w_5) \]
Split IF \[ \beta_s J_s \geq \tau_1 \]

Criterion 2
\[ \gamma_1 = (w_2)^2 + (w_3)^2 \]
\[ \gamma_2 = (w_4)^2 + (w_5)^2 \]
\[ \gamma_m = \max(\gamma_1, \gamma_2) \]

\[ \gamma_1 \leq \tau_2 \cdot \gamma_m \leq \gamma_2 \]
\[ \gamma_{1,2} \geq \tau_2 \cdot \gamma_m \]
\[ \gamma_2 \leq \tau_2 \cdot \gamma_m \leq \gamma_1 \]
Interpolation of Non-Smooth Functions

Evolution of interpolated function, random elements and corresponding sparse grids using adaptive interpolation procedure

\[ [N_d = 19, N = 95] \quad [N_d = 97, N = 485] \quad [N_d = 177, N = 885] \]
Interpolation of Non-Smooth Functions

Interpolation error using classical and adaptive construction

Error in Mean:
Classical (N=69633) ~ 5e-03
Adaptive (N=1425) ~ 2e-04

Relative error in mean (top) and variance (bottom) using classical and adaptive construction.
Static Analysis of MEMS Switch

Represent uncertain parameters in terms of independent random variables

\[ E = E_0 [1 + 0.1(2\xi_1 - 1)] \]
\[ g = g_0 [1 - 0.1(2\xi_2 - 1)] \]

Statistics for vertical tip displacement \([V=7.0\ V]\)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Nominal</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus (E) [GPa]</td>
<td>169</td>
<td>10%</td>
</tr>
<tr>
<td>Gap (g) [\mu m]</td>
<td>1.0</td>
<td>10%</td>
</tr>
</tbody>
</table>

Statistics for vertical tip displacement \([V=7.0\ V]\)

<table>
<thead>
<tr>
<th></th>
<th>MC (N=10,000)</th>
<th>GPC (p=3)</th>
<th>SC (N=135)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean [\mu m]</td>
<td>-0.1381</td>
<td>-0.1381</td>
<td>-0.1381</td>
</tr>
<tr>
<td>Std [\mu m]</td>
<td>0.0299</td>
<td>0.0293</td>
<td>0.0294</td>
</tr>
</tbody>
</table>
Pull-In Instability

Vertical tip displacement as a function of Young’s modulus and gap \([V=10.0 \text{ V}]\).

Performance parameters
- Vertical tip deflection at applied voltage \(V\)
- Pull-in voltage \(V_p\)

Adaptively refined random domain and sparse grid

Table: Pull-in voltage and tip deflection for MEMS switch using deterministic code

<table>
<thead>
<tr>
<th>Sample</th>
<th>(E/E_0)</th>
<th>(g/g_0)</th>
<th>Pull-in voltage</th>
<th>Vertical tip displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>S0</td>
<td>1.0</td>
<td>1.0</td>
<td>9.2</td>
<td>-0.3431</td>
</tr>
<tr>
<td>S1</td>
<td>0.9</td>
<td>0.9</td>
<td>7.3</td>
<td>-0.3007</td>
</tr>
<tr>
<td>S2</td>
<td>1.1</td>
<td>1.1</td>
<td>11.4</td>
<td>-0.4217</td>
</tr>
</tbody>
</table>
Stochastic Pull-In Curve

Worst-case and mean pull-in behavior of the MEMS switch.
Dynamic Analysis of MEMS Switch

Objectives:
- Quantify the effect of uncertain parameters on the performance parameters
- Identify critical design parameters

Electrostatically actuated MEMS switch [McCarthy et al. JMEMS, 2002]

**Table: Uncertain design parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Nominal</th>
<th>Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus (E) [GPa]</td>
<td>207</td>
<td>10%</td>
</tr>
<tr>
<td>Beam length (L) [μm]</td>
<td>70.0</td>
<td>2%</td>
</tr>
<tr>
<td>Beam thickness (t) [μm]</td>
<td>2.0</td>
<td>10%</td>
</tr>
<tr>
<td>Gap (g) [μm]</td>
<td>1.5</td>
<td>10%</td>
</tr>
<tr>
<td>Drive electrode (x) [μm]</td>
<td>35.0</td>
<td>2%</td>
</tr>
</tbody>
</table>

**Performance parameters:**
- Vertical tip displacement $Y_s$ (V=100 V)
- Time taken to strike the drain $T_s$ (V=200 V)

Input parameters: $\alpha = [E, L, t, g, x_g]$

Output parameters: $\beta = [Y_s, T_s]$

$\beta = f(\alpha)$

Number of stochastic dimensions $n=5$. 
Verification using Monte Carlo (MC)

Vertical tip deflection as a function of beam thickness and gap

<table>
<thead>
<tr>
<th>Vertical deflection</th>
<th>Strike time</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC (2000)</td>
<td>SC (61)</td>
</tr>
<tr>
<td>MC (2000)</td>
<td>SC (241)</td>
</tr>
<tr>
<td>Mean</td>
<td>Mean</td>
</tr>
<tr>
<td>-0.1712</td>
<td>0.8676</td>
</tr>
<tr>
<td>-0.1735</td>
<td>0.8619</td>
</tr>
<tr>
<td>Std</td>
<td>Std</td>
</tr>
<tr>
<td>0.0446</td>
<td>0.08</td>
</tr>
<tr>
<td>0.0450</td>
<td>0.0699</td>
</tr>
</tbody>
</table>

PDF of vertical tip deflection

PDF of strike time
Sensitivity Analysis

**Vertical tip deflection**

- Young's modulus
- Length
- Thickness
- Gap
- Location of drive electrode

**Strike time**

- Young's modulus
- Length
- Thickness
- Gap
- Location of drive electrode

**Observations:**

- Explicit expression for the output parameters in terms of design variables $\beta = f(\alpha)$
- Variations in beam thickness and gap leads to maximum variability in the switch performance
- Beam length has negligible effect on deflection, and variation in Young’s modulus is insignificant for strike time
Design under Uncertainties

An illustration of design under uncertainties: PDFs of output parameters and acceptable range of variation (shaded region) for various levels of input uncertainty.

Acceptable range of variation

\[-0.25 \, \mu m \leq Y_s \leq -0.11 \, \mu m\]

\[T_s \leq 1.0 \, \mu s\]

Restricting beam thickness and gap to within 5% of nominal values satisfies the design criterion.
Summary

- Developed an adaptive stochastic collocation framework, based on sparse grid interpolation, suitable for discontinuities in random domain
  - Provides high resolution as well as easy implementation
  - Significant reduction in computational effort as compared to classical Smolyak construction, in the presence of discontinuities
  - Can be effectively used to quantify the effect of uncertainty on device performance and to identify critical design parameters
- Stochastic adaptive collocation techniques will be implemented in MEMOSA